SOLUTIONS - EXERCISE 1

(a) We have for the **joint distribution**

$$\begin{aligned}
P(U=1,R=1) &= \frac{1}{3}\cdot\frac{80}{100}=0.267\\
P(U=1,R=0) &= \frac{1}{3}\cdot\frac{20}{100}=0.067\\
P(U=2,R=1) &= \frac{1}{3}\cdot\frac{100}{100}=0.333\\
P(U=2,R=0) &= \frac{1}{3}\cdot\frac{0}{100}=0.000\\
P(U=3,R=1) &= \frac{1}{3}\cdot\frac{10}{100}=0.033\\
P(U=3,R=0) &= \frac{1}{3}\cdot\frac{90}{100}=0.300
\end{aligned}$$

(b) And the **marginal distributions** are

$$P(U=i)=\frac{1}{3}\quad (i=1,2,3)$$

$$P(R=1)=0.267+0.333+0.033=0.633 \ \text{ and } \ P(R=0)=1-0.633=0.367$$

(c) The **conditional distribution** is:

$$\begin{aligned}
P(U=1|R=1) &= \frac{0.267}{0.633}=0.422\\
P(U=2|R=1) &= \frac{0.333}{0.633}=0.526\\
P(U=3|R=1) &= \frac{0.033}{0.633}=0.052
\end{aligned}$$

SOLUTIONS - EXERCISE 2

$$\begin{aligned}
p(y) &= \int p(y|\theta)\cdot p(\theta)\ d\theta\\
&= \int \frac{\theta^y\cdot e^{-\theta}}{\Gamma(y+1)}\cdot\lambda\cdot e^{-\lambda\theta}\ d\theta\\
&= \lambda\cdot\int\frac{\theta^y}{\Gamma(y+1)}\cdot e^{-\theta(\lambda+1)}\ d\theta\\
&= \lambda\cdot\left(\frac{1}{\lambda+1}\right)^{y+1}\cdot\int\frac{(\lambda+1)^{y+1}}{\Gamma(y+1)}\cdot\theta^y\cdot e^{-\theta(\lambda+1)}\ d\theta\\
&= \lambda\cdot\left(\frac{1}{\lambda+1}\right)^{y+1}\cdot 1\\
&= \left(\frac{\lambda}{1+\lambda}\right)^{y}\cdot\frac{\lambda}{1+\lambda}
\end{aligned}$$

SOLUTIONS - EXERCISE 3

$$
\begin{aligned}
p_\tau(y_1,\dots,y_n|b_1) &= \left(\prod_{i=1}^n \frac{b_1^{a_1}}{\Gamma(a_1)}\cdot y_i^{a_1-1}\cdot \exp\{-b_1\cdot y_i\}\right)^\tau \\
&= \prod_{i=1}^n \left(\frac{b_1^{a_1}}{\Gamma(a_1)}\right)^\tau \cdot y_i^{\tau(a_1-1)}\cdot \exp\{-b_1\cdot\tau\cdot y_i\} \\
&= \left(\frac{b_1^{a_1}}{\Gamma(a_1)}\right)^{n\cdot\tau}\cdot\left(\prod_{i=1}^n y_i^{\tau(a_1-1)}\right)\cdot \exp\{-b_1\cdot\tau\cdot\sum_{i=1}^n y_i\}
\end{aligned}
$$

and

$$
p(b_1) = \frac{b_2^{a_2}}{\Gamma(a_2)}\cdot b_1^{a_2-1}\cdot\exp\{-b_2\cdot b_1\}
$$

For the power posterior (function of b_1) we have:

$$
\begin{aligned}
p_\tau(b_1|y_1,\dots,y_n) &\propto p_\tau(y_1,\dots,y_n|b_1)\cdot p(b_1) \\
&\propto b_1^{a_1\cdot n\cdot\tau}\cdot\exp\{-b_1\cdot\tau\cdot\sum_{i=1}^n y_i\}\cdot b_1^{a_2-1}\cdot\exp\{-b_2\cdot b_1\} \\
&\propto b_1^{a_2+a_1\cdot n\cdot\tau-1}\cdot\exp\{-b_1\cdot(b_2+\tau\cdot\sum_{i=1}^n y_i)\}
\end{aligned}
$$

We recognize the characteristic shape of a Gamma density and conclude for the power posterior:

$$
b_1|(Y_1,\dots,Y_n,\tau)\sim \text{GAM}\left(a_2+a_1\cdot n\cdot\tau,\ b_2+\tau\cdot\sum_{i=1}^n y_i\right)
$$

SOLUTIONS - EXERCISE 4

We have for the posterior:

$$
\begin{aligned}
p(\mathbf{V}|\mathbf{W}_1,\dots,\mathbf{W}_n) &\propto p(\mathbf{W}_1,\dots,\mathbf{W}_n|\mathbf{V})\cdot p(\mathbf{V}) \\
&\propto \left(\prod_{i=1}^n p(\mathbf{W}_i|\mathbf{V})\right)\cdot p(\mathbf{V})
\end{aligned}
$$

As a function of $\mathbf{V}$ we have

$$
\begin{aligned}
p(\mathbf{W}_i|\mathbf{V}) &\propto \det(\mathbf{V})^{\alpha/2}\cdot\exp\left(-\frac{1}{2}tr(\mathbf{V}\mathbf{W}_i)\right) \\
p(\mathbf{V}) &\propto \det(\mathbf{V})^{(\alpha_2-S-1)/2}\cdot\exp\left(-\frac{1}{2}tr(\mathbf{V}_0\mathbf{V})\right)
\end{aligned}
$$

It follows (as a function of $\mathbf{V}$):

$$\begin{aligned} p(\mathbf{W}_1, \ldots, \mathbf{W}_n | \mathbf{V}) &\propto \prod_{i=1}^{n} \det(\mathbf{V})^{\alpha/2} \cdot \exp\left(-\frac{1}{2} tr(\mathbf{V}\mathbf{W}_i)\right) \\ &\propto \det(\mathbf{V})^{n\cdot\alpha/2} \cdot \exp\left(-\frac{1}{2} tr\left(\mathbf{V}\left(\sum_{i=1}^{n} \mathbf{W}_i\right)\right)\right) \\ &\propto \det(\mathbf{V})^{n\cdot\alpha/2} \cdot \exp\left(-\frac{1}{2} tr\left(\left(\sum_{i=1}^{n} \mathbf{W}_i\right) \cdot \mathbf{V}\right)\right) \end{aligned}$$

Henceforth

$$\begin{aligned} p(\mathbf{V} | \mathbf{W}_1, \ldots, \mathbf{W}_n) &\propto \det(\mathbf{V})^{n\cdot\alpha/2} \cdot \exp\left(-\frac{1}{2} tr\left(\left(\sum_{i=1}^{n} \mathbf{W}_i\right) \cdot \mathbf{V}\right)\right) \\ &\quad \cdot \det(\mathbf{V})^{(\alpha_2 - S - 1)/2} \cdot \exp\left(-\frac{1}{2} tr(\mathbf{V}_0 \mathbf{V})\right) \\ &\propto \det(\mathbf{V})^{(n\cdot\alpha + \alpha_2 - S - 1)/2} \cdot \exp\left(-\frac{1}{2} tr\left(\left(\mathbf{V}_0 + \sum_{i=1}^{n} \mathbf{W}_i\right) \cdot \mathbf{V}\right)\right) \end{aligned}$$

We recognize the characteristic shape of a Wishart density and conclude for the posterior:

$$\mathbf{V} | (\mathbf{W}_1, \ldots, \mathbf{W}_n) \sim \mathcal{W}\left(\alpha_2 + n \cdot \alpha_1,\ \mathbf{V}_0 + \sum_{i=1}^{n} \mathbf{W}_i\right)$$

SOLUTIONS - EXERCISE 4(b)

For dimension $S = 1$ we have:

$$\begin{aligned} p(\mathbf{W} | \mathbf{T}, \alpha) &= \frac{\det(\mathbf{W})^{(\alpha - 1 - 1)/2}}{Z(1, \mathbf{T}, \alpha)} \cdot \exp(-\frac{1}{2} tr(\mathbf{T}\mathbf{W})) \\ &= \frac{\mathbf{W}^{\frac{\alpha}{2} - 1}}{Z(1, \mathbf{T}, \alpha)} \cdot \exp(-\frac{1}{2} \mathbf{T}\mathbf{W}) \end{aligned}$$

where

$$\begin{aligned} Z(1, \mathbf{T}, \alpha) &= 2^{1\cdot\alpha/2} \cdot \pi^{1(1-1)/4} \cdot \det(\mathbf{T})^{-\alpha/2} \cdot \prod_{j=1}^{1} \Gamma\left(\frac{\alpha - j + 1}{2}\right) \\ &= 2^{\alpha/2} \cdot \mathbf{T}^{-\alpha/2} \cdot \Gamma\left(\frac{\alpha}{2}\right) \\ &= \left(\frac{2}{\mathbf{T}}\right)^{\alpha/2} \cdot \Gamma\left(\frac{\alpha}{2}\right) \end{aligned}$$

It follows:

$$p(\mathbf{W} | \mathbf{T}, \alpha) = \frac{\left(\frac{\mathbf{T}}{2}\right)^{\frac{\alpha}{2}}}{\Gamma\left(\frac{\alpha}{2}\right)} \cdot \mathbf{W}^{\frac{\alpha}{2} - 1} \cdot \exp(-\frac{\mathbf{T}}{2} \cdot \mathbf{W})$$

which is the density of the GAM($\frac{\alpha}{2}, \frac{\mathbf{T}}{2}$) distribution.

SOLUTIONS - EXERCISE 5

We have

$$\mathbf{Y} \sim \mathcal{N}_n(\mathbf{1} \cdot \mu, \sigma^2 \mathbf{I})$$

(a) It follows

$$\mathbf{Y}|\theta \sim \mathcal{N}_n(\mathbf{1} \cdot \theta, \sigma^2 \mathbf{I} + \mathbf{1}\sigma^2 \mathbf{1}^\mathsf{T}) = \mathcal{N}_n(\mathbf{1} \cdot \tau, \sigma^2(\mathbf{I} + \mathbf{1}_{n,n}))$$

(b) It follows from (a):

$$\mathbf{Y} \sim \mathcal{N}_n(\mathbf{1} \cdot 1, \sigma^2(\mathbf{I} + \mathbf{1}_{n,n}) + \mathbf{1}\sigma^2 \mathbf{1}^\mathsf{T}) = \mathcal{N}_n(\mathbf{1} \cdot 1, \sigma^2(\mathbf{I} + 2 \cdot \mathbf{1}_{n,n}))$$

(c) It follows from (b):

$$\text{VAR}(Y_1) = 3\sigma^2 \quad \text{and} \quad \text{COV}(Y_1, Y_2) = 2\sigma^2$$

(d) We get with the posterior rule:

$$\mathbf{\Sigma}_\dagger = \left(\sigma^{-2} + \mathbf{1}^\mathsf{T}(\sigma^2 \cdot \mathbf{I})^{-1}\mathbf{1}\right)^{-1} = \left(\sigma^{-2} + n\sigma^{-2}\right)^{-1} = \frac{\sigma^2}{n+1}$$

and so:

$$\mu|(\mathbf{Y}, \theta) \sim \mathcal{N}\left(\frac{\sigma^2}{n+1}\left(\mathbf{1}^\mathsf{T} \cdot (\sigma^2 \mathbf{I})^{-1} \cdot \mathbf{Y} + \sigma^{-2} \cdot \theta\right), \ \frac{\sigma^2}{n+1}\right) = \mathcal{N}\left(\frac{\sum_{i=1}^{n} Y_i + \theta}{n+1}, \ \frac{\sigma^2}{n+1}\right)$$

(e) We get with the posterior rule:

$$\mathbf{\Sigma}_\dagger = \left(\sigma^{-2} + 1^\mathsf{T}(\sigma^2)^{-1}1\right)^{-1} = \left(\sigma^{-2} + \sigma^{-2}\right)^{-1} = \frac{\sigma^2}{2}$$

and so:

$$\theta|\mu \sim \mathcal{N}\left(\frac{\sigma^2}{2}\left(1^\mathsf{T} \cdot (\sigma^2)^{-1} \cdot \mu + \sigma^{-2} \cdot 1\right), \ \frac{\sigma^2}{2}\right) = \mathcal{N}\left(\frac{\mu + 1}{2}, \ \frac{\sigma^2}{2}\right)$$